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RENORMALIZATION METHOD IN THE CASE OF AN UNBOUNDED MEDIUM IN THE ABSENCE OF EXTERNAL FORCES

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A solution is given of the problem of computing the fields and effective elastic moduli for an unbounded medium in the absence of external forces. By introducing renormalized fields and material characteristics, the interaction between inhomogeneity domains is separated into local and nonlocal components. Taking account of the former results in models and approximations yielding elastic fields and effective elastic moduli in the form obtained in the singular approximation. This latter is an extension of the known Voigt (homogeneous strains) and Reuss (homogeneous stresses) hypotheses. It is shown that nature of the equivalence noted is that the elastic fields in the models and approximations mentioned have the form of fields computed for an isolated. inclusion placed in an unbounded matrix in the absence of volume forces.

1. Let the inhomogeneous unbounded medium under consideration be characterized by an elastic modulus tensor of the fourth rank $\lambda(\mathbf{r})$. (Here and henceforth, the tensor indices are omitted for simplicity). In addition, let us introduce a comparison medium different from the initial medium in only the elastic properties which are described by the homogeneous tensor λ_c. However, the property of homogeneity of the comparison field is not absolute. Thus, for instance, it is expedient to select the field of a laminar medium without imperfections for which the problem is solved exactly, as λ_c in solving an elastic problem for a laminar medium with small imperfections in the layers [1].

The fields of the displacements $\mathbf{u}$ and $\mathbf{u}_c$, corresponding to both media satisfy equations [2,3]

$$L\mathbf{u} = 0,\ L = \operatorname{div} \lambda \operatorname{def} \tag{1.1}$$
$$L_c\mathbf{u}_c = 0,\ L_c = \operatorname{div} \lambda_c \operatorname{def}$$

where the operator def connects the strain tensor ε and the displacement vector $\mathbf{u}$ by means of the equality $\varepsilon = \operatorname{def} \mathbf{u}$.

In contrast to problems in the classical theory of the elasticity of homogeneous media, an additional problem of determining the effective elastic constants occurs in statistical elasticity theory. These constants enter Hooke's law, averaged over the ensemble of realizations. Averaging over the volume and over the ensemble of realizations yields an identical result for unbounded media in the absence of external forces, hence, in this case the effective elastic moduli are also the macroscopic parameters of the medium. No distinction is made below between the two averaging methods.

It is convenient to find the fields of the stresses σ and the strains ε and the effective elastic moduli λ_* on the basis of (1.1) by introducing certain renormalized quantities τ, e and l related to σ, ε and λ by the equalities [2,3]

$$\tau = le = \sigma - \lambda_c \varepsilon, \qquad e = (I - g\lambda')\,\varepsilon \tag{1.2}$$
$$l = \lambda' (I - g\lambda')^{-1}, \ \lambda' = \lambda - \lambda_c$$

where g is a certain constant tensor determining the intensity of the singular component of the second derivative of the Green's tensor G of (1.1) for the comparison medium.
Solving (1.1) in terms of τ, e, l, we obtain [2, 3]

$$e = R\langle e\rangle, \quad l_* = \langle lR\rangle, \qquad \langle R\rangle = I \tag{1.3}$$

where R is a certain nonlocal operator defined on the basis of the formal component of the second derivative of the Green's tensor. The solution in the form (1.3) is convenient in that local interactions are separated from nonlocal interactions therein.
In the majority of papers devoted to this question, one is usually limiting oneself to approximate solutions. The best are those which take account of local interactions of all orders completely and exactly. It can be shown [4] that all approaches of such a kind lead to the analytically identical results and distinctions between them can only occur because of the distinctions in the parameters λ_c used. This approximation is called singular (the S- approximation).
Assuming $R_s = I$ in the S- approximation, we write

$$l_* = \langle l\rangle, \quad e = \langle e\rangle \tag{1.4}$$

The solution (1.4) shows that the approximations ignoring nonlocal interactions, or equivalently, the formal component of the second derivative of the Green's tensor G, are intermediate between the Voigt $\varepsilon = \langle\varepsilon\rangle$ and the Reuss $\sigma = \langle\sigma\rangle$ approximations. Indeed, by introducing the tensor b_c thus $g(\lambda_c + b_c) = -I$ and using the relation (1.2) between e and ε, we obtain from (1.4)

$$\sigma + b_c\varepsilon = \langle\sigma + b_c\varepsilon\rangle = \langle\sigma\rangle + b_c\langle\varepsilon\rangle \tag{1.5}$$

Hence, as $b_c \to 0$ and $b_c \to \infty$ we arrive at the Reuss and Voigt approximations, respectively.

2. Let us show that the solution in the form (1.4), (1.5) agrees with the solution obtained for one inclusion in a matrix.
Let us make a preliminary remark. A derivative of homogeneous degree -2 of a function Φ having a singularity at zero can be determined as follows in the theory of generalized functions [5]:

$$\Phi_{,i} = \Phi_{,i}^{s} + \Phi_{,i}^{f}, \quad \int_V \Phi_{,i}^{s}(\mathbf{r})\,\varphi(\mathbf{r})\,d\mathbf{r} = \varphi(0)\int_{S_0} \Phi\, dS_i \tag{2.1}$$
$$\int_V \Phi_{,i}^{f}(\mathbf{r})\,\varphi(\mathbf{r})\,d\mathbf{r} = \int_{V_0} \Phi_{,i}(\mathbf{r})\,[\varphi(\mathbf{r}) - \varphi(0)]\,d\mathbf{r} + \int_{V-V_0} \Phi_{,i}(\mathbf{r})\,\varphi(\mathbf{r})\,d\mathbf{r}$$

where φ is the auxiliary function, S_0 is the surface of the volume V_0 including the point of local nonintegrability $r = 0$. It can be shown [5] that the surface integral depends only on the shape of S_0, but not on the magnitude of V_0.
We have from (2.1) (δ is the delta function)

$$\Phi,_i^s = \delta \int\limits_{S_0} \Phi \, dS_i \tag{2.2}$$

Taking the above into account, we introduce the notation

$$(\text{def div}^T)^s \, G = g\delta, \quad \text{div}^T \, G * \equiv G * \ \text{div} \ (\text{def div}^T)^f \, G * F = hF$$

where s and f denote the singular and formal parts, and T is the transpose. In the case under consideration of an unbounded medium and no external forces, the solution (1.3) has the form [2,3]

$$R = \sum_0^\infty (hl)^n, \quad \langle e \rangle = \text{const} \tag{2.3}$$

The solution (2.3), valid for a medium with arbitrary distribution of inhomogeneities, is evidently applicable even in a particular case of a medium in the form of a matrix with an inclusion.

Describing the elastic properties of the inclusion and the matrix by the constant tensors λ_i and λ_c, let us note that the tensor l is not zero only in the inclusion. Consequently, the integration associated with the operator h will be only limited to the volume of the inclusion.

Taking the above into account and introducing the following notation in conformity with (2.1)

$$\Phi,_i (\mathbf{r}) \equiv \text{def div}^T \, G \, (\mathbf{r}), \quad \varphi \, (r) \equiv l^T \otimes l \langle e \rangle$$

where $\otimes$ denotes the direct product of the tensors, and the origin is in the inclusion, we represent the quantity $z = lhl \, \langle e \rangle$ in the form

$$z = \int\limits_V \Phi,_i^f (\mathbf{r}) \, \varphi \, (\mathbf{r}) \, d\mathbf{r} = \int\limits_{V_0} \Phi,_i (\mathbf{r}) \, [\varphi \, (r) - \varphi \, (0)] \, d\mathbf{r} = 0$$

This equality is valid because of the constancy of $\varphi \, (\mathbf{r})$ within V_0. Therefore, terms of the series (2.3) of the form $(hl)^n \, \langle e \rangle$ vanish everywhere for $n \geqslant 2$, and only for the points $\mathbf{r} \in V_0$ for $n = 1$.

Therefore, the field $e \, (\mathbf{r})$ has the form

$$e = R \langle e \rangle, \quad R = \begin{cases} I, & \mathbf{r} \in V_0 \\ I + hl, & \mathbf{r} \in V - V_0 \end{cases} \tag{2.4}$$

We find analogously from (1.3)

$$l_* = \langle lR \rangle = v_i l_i \equiv \langle l \rangle, \quad v_i = V_0 / V \tag{2.5}$$

As is seen from (2.4), the field is homogeneous in the inclusion but the field in the matrix contains the term $hl \, \langle e \rangle$ defining the inhomogeneous component. However, since $l = 0$ in the matrix, the field in the matrix is not used in the computation of l_*, and therefore, this inhomogeneity is not manifest.

Let us also note that (2.4) and (2.5), which agree with the solution (1.4) in the S-approximation, and therefore reduce to the fields of ε, σ, λ_* satisfying (1.4) and

(1.5), are valid in contrast to the latters as $v_i \to 0$.

The solution presented, which is based on using the renormalization method on the one hand, explains the nature of the equivalence of the approximate solutions based on taking account only the local interactions. On the other hand, it indicates still un-utilized possibilities of the method.

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